Physics 523, General Relativity

Homework 4

Due Wednesday, 25th October 2006

Jacob Lewis Bourjaily

Problem 1

Recall that the worldline of a continuously accelerated observer in flat space relative to some inertial frame can be described by

$$t(\lambda,\alpha)=\alpha\sinh(\lambda)\qquad\text{and}\qquad x(\lambda,\alpha)=\alpha\cosh(\lambda),\tag{1.a.1}$$

where λ is an affine parameter of the curve with $\alpha\lambda$ its proper length—i.e. the 'time' as measured by an observer in the accelerated frame. Before, we considered α to be constant and only varied λ. We are now going to consider the entire (non-surjective) curvilinear map from two-dimensional Minkowski to-space to itself defined by equation (1.a.1).

a) Consider the differential map from t,x-coordinate charts to λ,α-coordinate charts implied by equation (1.a.1)—lines of constant α are in the λ-direction, and lines of constant λ are in the α-direction. We are to show that wherever lines of constant α meet lines of constant λ, the two curves are orthogonal.

To show that the two curves cross 'orthogonally,' we must demonstrate that their tangent vectors are orthogonal at points of intersection. This is not particularly hard. Because orthogonality is a frame independent notion, we may as well compute this in t,x-space. The lines of constant λ parameterized by α are given by $\ell(\alpha)=(\alpha\sinh\lambda,\alpha,\cosh\lambda)$, which has the associated tangent vector

$$\vec{\ell}\equiv\frac{\partial\ell(\alpha)}{\partial\alpha}=(\sinh\lambda,\cosh\lambda)\,.\tag{1.a.2}$$

Similarly, lines of constant α parameterized by λ are $\vartheta(\lambda)=(\alpha\sinh\lambda,\alpha\cosh\lambda)$, which obviously has the associated tangent

$$\vec{\vartheta}\equiv\frac{\partial\vartheta(\lambda)}{\partial\lambda}=(\alpha\cosh\lambda,\alpha\sinh\lambda)\,.\tag{1.a.3}$$

We see at once that

$$g(\vec{\ell},\vec{\vartheta})=-\alpha\cosh\lambda\sinh\lambda+\alpha\sinh\lambda\cosh\lambda=0.\tag{1.a.4}$$

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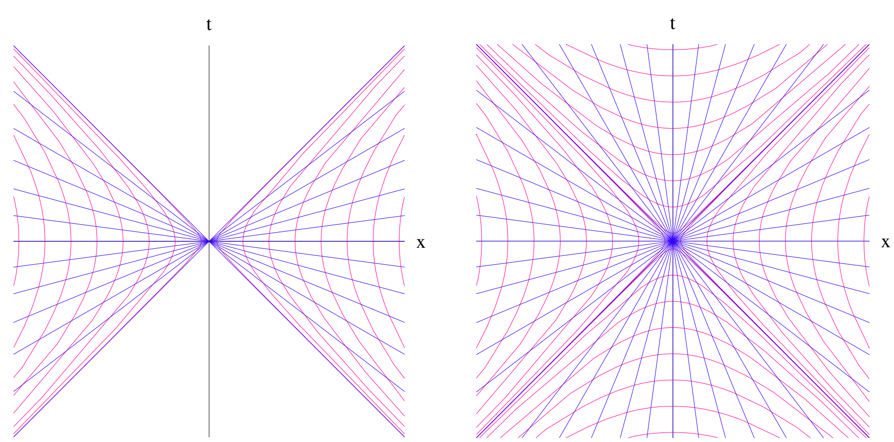

Figure 1. The orthogonal curvilinear coordinate charts which could be used by a uniformly accelerated observer in Minkowski spacetime. The red curves indicate surfaces of constant α and the blue curves indicate surfaces of constant λ. The diagram on the left shows the coordinate patch explicitly constructed in Problem 1, and the diagram on the right extends this construction to the whole of Minkowski space—minus lightcone of an observer at the origin.

b) We are to show that the map specified by equation (1.a.1) gives rise to an orthogonal coordinate system that covers half of Minkowski space in two disjoint patches. We should also represent this coordinate system diagrammatically.

From our work in part (a) above, we know that the tangent vectors to the lines of constant λ and α are given by

$$\vec{e}_\lambda = \alpha \cosh\lambda \ \vec{e}_t + \alpha \sinh\lambda \ \vec{e}_x \qquad \text{and} \qquad \vec{e}_\alpha = \sinh\lambda \ \vec{e}_t + \cosh\lambda \ \vec{e}_x. \tag{1.b.1}$$

Therefore, the differential map (where greek letters are used to indicate λ, α-coordinates) is given by

$$\Lambda^\mu{}_n = \begin{pmatrix} \alpha\cosh\lambda & \alpha\sinh\lambda \\ \sinh\lambda & \cosh\lambda \end{pmatrix}. \tag{1.b.2}$$

We see immediately that the Jacobian, $\det(\Lambda) = \alpha \neq 0$ which implies that the λ, α coordinate system is good generically (where it is defined). That it is 'orthogonal' is manifest because $\vec{e}_\lambda \cdot \vec{e}_\alpha = 0$ by part (a) above.

Note that the charts of (1.a.1) are not well-defined on or within the past or future lightcones of an observer at the origin: the curves of $\alpha =$ constant, the hyperbolas, are all time-like and outside the past and future lightcones of an observer at the origin; and the lines of $\lambda =$ constant are all spacelike and coincident at the origin. It does not take much to see that these coordinates have no overlap within the past and future lightcones of the Minkowski origin.

The coordinate system spanned by λ, α is shown in Figure 1.

c) We are to find the metric tensor and its associated Christoffel symbols of the coordinate charts described above.

Using equation (1.b.1), we can directly compute the components of the metric tensor in λ, α coordinates—$g_{\mu\nu} = g(\vec{e}_\mu, \vec{e}_\nu)$ where λ is in the '0'-position—

$$g_{\mu\nu} = \begin{pmatrix} -\alpha^2 & 0 \\ 0 & 1 \end{pmatrix}. \tag{1.c.1}$$

The Christoffel symbols can be computed by hand rather quickly in this case; but we will still show some rough steps. Recall that the components of the Christoffel symbol $\Gamma^\mu_{\nu\rho}$ are given by

$$\Gamma^\mu_{\nu\rho}\vec{e}_\mu = \left(\frac{\partial \vec{e}_\nu}{\partial x^\rho}\right)^\mu \vec{e}_\mu.$$

Again making use of equation (1.b.1), we see that

$$\frac{\partial \vec{e}_\alpha}{\partial \alpha} = 0 \qquad \Longrightarrow \qquad \Gamma^\alpha_{\alpha\alpha} = \Gamma^\lambda_{\alpha\alpha} = 0. \tag{1.c.2}$$

Slightly less trivial, we see

$$\frac{\partial \vec{e}_\alpha}{\partial \lambda} = \cosh\lambda \ \vec{e}_t = \sinh\lambda \ \vec{e}_x = \frac{1}{a}\vec{e}_\lambda \qquad \Longrightarrow \qquad \Gamma^\lambda_{\alpha\lambda} = \Gamma^\lambda_{\lambda\alpha} = \frac{1}{\alpha}; \tag{1.c.3}$$

and,

$$\frac{\partial \vec{e}_\lambda}{\partial \lambda} = \alpha \sinh\lambda \ \vec{e}_t + \alpha\cosh\lambda \vec{e}_x = \alpha \vec{e}_\alpha \qquad \Longrightarrow \qquad \Gamma^\alpha_{\lambda\lambda} = \alpha, \quad \text{and} \quad \Gamma^\lambda_{\lambda\lambda} = 0. \tag{1.c.4}$$

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Problem 2

We are to find the Lie derivative of a tensor whose components are $T^{ab}{}_c$.

Although we are tempted to simply state the result derived in class and found in numerous textbooks, we will at least feign a derivation. Let us begin by recalling that the components of the tensor $\mathbf{T}$ are given by $T^{ab}{}_c = \mathbf{T}(\mathbf{E}^a, \mathbf{E}^b, \mathbf{E}_c)$ where the $\mathbf{E}$'s are basis vector- and one-form fields. Now, by the Leibniz rule for the Lie derivative we know that

$$\pounds_X\big(\mathbf{T}(\mathbf{E}^a,\mathbf{E}^b,\mathbf{E}_c)\big) = \pounds_X(T)\left(\mathbf{E}^a,\mathbf{E}^b,\mathbf{E}_c\right) + \mathbf{T}\left(\pounds_X(\mathbf{E}^a),\mathbf{E}^b,\mathbf{E}_c\right) + \mathbf{T}\left(\mathbf{E}^a,\pounds_X(\mathbf{E}^b),\mathbf{E}_c\right) + \mathbf{T}\left(\mathbf{E}^a,\mathbf{E}^b,\pounds_X(\mathbf{E}_c)\right). \tag{2.a.1}$$

Now, the first term on the right hand side of equation (2.a.1) gives the components of $\pounds_X(\mathbf{T})$, which is exactly what we are looking for. Rearranging equation (2.a.1) and converting our notation to components, we see

$$\left(\pounds_X(\mathbf{T})\right)^{ab}{}_c = \pounds_X\big(T^{ab}{}_c\big) - T^{\alpha b}{}_c\left(\pounds_X(\mathbf{E}^a)\right)_\alpha - T^{a\beta}{}_c\left(\pounds_X(\mathbf{E}^b)\right)_\beta - T^{ab}{}_\gamma\left(\pounds_X(\mathbf{E}_c)\right)^\gamma. \tag{2.a.2}$$

Now, we can either use some identities or just simply recall that

$$\left(\pounds_X(\mathbf{E}^a)\right)_\alpha = \frac{\partial X^a}{\partial x^\alpha} \qquad \text{and} \qquad \left(\pounds_X(\mathbf{E}_c)\right)^\gamma = \frac{\partial X^\gamma}{\partial x^c}. \tag{2.a.3}$$

We now have all the ingredients; putting everything together, we have

$$\left(\pounds_X(\mathbf{T})\right)^{ab}{}_c = X^\delta\frac{\partial}{\partial x^\delta}\left(T^{ab}{}_c\right) - T^{\alpha b}{}_c\frac{\partial X^a}{\partial x^\alpha} - T^{a\beta}{}_c\frac{\partial X^b}{\partial x^\beta} + T^{ab}{}_\gamma\frac{\partial X^\gamma}{\partial x^c}. \tag{2.a.4}$$

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Problem 3

Theorem: Acting on any tensor $\mathbf{T}$, the Lie derivative operator obeys

$$\pounds_U\pounds_V(\mathbf{T}) - \pounds_V\pounds_U(\mathbf{T}) = \pounds_{[U,V]}(\mathbf{T}). \tag{3.a.1}$$

proof: We will proceed by induction. Let us suppose that the theorem holds for all tensors of rank less than or equal to $\binom{r}{s}$ for some $r, s \geq 1$. We claim that this is sufficient to prove the hypothesis for any tensor of rank $\binom{r}{s+1}$ or $\binom{r+1}{s}$. (The induction argument is identical for the two cases—our argument will depend on which index is advancing—so it is not necessary to expound both cases.)

Now, all rank $\binom{r+1}{s}$ tensors can be written as *a sum of* tensor products between $\binom{r}{s}$ rank tensors $\mathbf{T}$ indexed by i and rank $\binom{1}{0}$ tensors $\mathbf{E}$, again indexed by i[1]. That is, we can express an arbitrary $\binom{r+1}{s}$ tensor as a sum of $\sum_i \mathbf{T}_i\otimes\mathbf{E}_i$—where i is just an index label! But this complication is entirely unnecessary: by the linearity of the Lie derivative, it suffices to show the identity for any one tensor product in the sum.

Making repeated use of the linearity of the Lie derivative and the Leibniz rule, we see

$$\begin{aligned}
\left(\pounds_U\pounds_V - \pounds_V\pounds_U\right)(\mathbf{T}\otimes\mathbf{E}) =& \pounds_U\Big(\left(\pounds_V\mathbf{T}\right)\otimes\mathbf{E} + \mathbf{T}\otimes\pounds_V\mathbf{E}\Big) - \pounds_V\Big(\left(\pounds_U\mathbf{T}\right)\otimes\mathbf{E} + \mathbf{T}\otimes\pounds_U\mathbf{E}\Big),\\
=& \Big(\pounds_U\pounds_V\mathbf{T}\Big)\otimes\mathbf{E} + \pounds_V\mathbf{T}\otimes\pounds_U\mathbf{E} + \pounds_U\mathbf{T}\otimes\pounds_V\mathbf{E} + \mathbf{T}\otimes\Big(\pounds_U\pounds_V\mathbf{E}\Big)\\
&- \Big(\pounds_V\pounds_U\mathbf{T}\Big)\otimes\mathbf{E} - \pounds_U\mathbf{T}\otimes\pounds_V\mathbf{E} - \pounds_V\mathbf{T}\otimes\pounds_U\mathbf{E} - \mathbf{T}\otimes\Big(\pounds_V\pounds_U\mathbf{E}\Big),\\
=&\Big(\pounds_{[U,V]}\mathbf{T}\Big)\otimes\mathbf{E} + \mathbf{T}\otimes\Big(\pounds_{[U,V]}\mathbf{E}\Big),\\
=&\pounds_{[U,V]}\Big(\mathbf{T}\otimes\mathbf{E}\Big);
\end{aligned}$$

where in the second to last line we used the induction hypothesis—applicable because both $\mathbf{T}$ and $\mathbf{E}$ are of rank $\binom{r}{s}$ or less.

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[1]The savvy reader knows that an arbitrary $\binom{r}{s}$ tensor can not be written as a tensor product of r contravariant and s covariant pieces; *however* every $\binom{r}{s}$ tensor can be written as a *sum of* such tensor products: indeed, this is exactly what is done when writing 'components' of the tensor.

It now suffices to show that the identity holds for all $\binom{0}{1}$ forms and all $\binom{1}{0}$ tensors[2]. We will actually begin one-step lower and note that equation (3.a.1) follows trivially from the Leibniz rule for 0-forms. Indeed, we see that for any 0-form f,

$$\begin{aligned}\pounds_U(\pounds_V f) &= \Big(\pounds_U V\Big) f + V\Big(\pounds_U f\Big),\\ &= \pounds_{[U,V]} f + \pounds_V(\pounds_U f)\,,\\ \therefore \Big(\pounds_U\pounds_V - \pounds_V\pounds_U\Big) f &= \pounds_{[U,V]} f. \end{aligned} \tag{3.a.2}$$

Now, to finish our proof, we claim that the identity holds for any $\binom{1}{0}$ and $\binom{0}{1}$ tensors, say $\mathbf{X}$ and $\mathbf{Y}$, respectively. Recall that a one form $\mathbf{Y}$ is a function mapping vector fields into scalars—i.e. $\mathbf{Y}(\mathbf{X})$ is a 0-form. For our own convenience, we will write $\mathbf{Y}(\mathbf{X}) \equiv \langle \mathbf{X}, \mathbf{Y}\rangle$. From our work immediately above, we know the identity holds for $\langle \mathbf{X}, \mathbf{Y}\rangle$:

$$\Big(\pounds_U\pounds_V - \pounds_V\pounds_U\Big)\langle \mathbf{X}, \mathbf{Y}\rangle = \pounds_{[U,V]}\Big(\langle \mathbf{X}, \mathbf{Y}\rangle\Big). \tag{3.a.3}$$

Because the Leibniz rule obeys contraction, we can expand out the equation above similar to as before. Indeed, almost copying the equations above verbatim we find

$$\begin{aligned}(\pounds_U\pounds_V - \pounds_V\pounds_U)(\langle \mathbf{X}, \mathbf{Y}\rangle) =& \pounds_U\Big(\langle(\pounds_V\mathbf{X}), \mathbf{Y}\rangle + \langle \mathbf{X}, \pounds_V\mathbf{Y}\rangle\Big) - \pounds_V\Big(\langle(\pounds_U\mathbf{X}), \mathbf{Y}\rangle + \langle \mathbf{X}, \pounds_U\mathbf{Y}\rangle\Big),\\ =&\quad \langle \pounds_U\pounds_V\mathbf{X}, \mathbf{Y}\rangle + \langle \pounds_V\mathbf{X}, \pounds_U\mathbf{Y}\rangle + \langle \pounds_U\mathbf{X}, \pounds_V\mathbf{Y}\rangle + \langle \mathbf{X}, \pounds_U\pounds_V\mathbf{Y}\rangle\\ &- \langle \pounds_V\pounds_U\mathbf{X}, \mathbf{Y}\rangle - \langle \pounds_U\mathbf{X}, \pounds_V\mathbf{Y}\rangle - \langle \pounds_V\mathbf{X}, \pounds_U\mathbf{Y}\rangle - \langle \mathbf{X}, \pounds_V\pounds_U\mathbf{Y}\rangle\,,\\ =& \Big\langle\Big(\pounds_U\pounds_V - \pounds_V\pounds_U\Big)\mathbf{X}, \mathbf{Y}\Big\rangle + \Big\langle \mathbf{X}, \Big(\pounds_U\pounds_V - \pounds_V\pounds_U\Big)\mathbf{Y}\Big\rangle\,,\\ =& \big\langle \pounds_{[U,V]}\mathbf{X}, \mathbf{Y}\big\rangle + \big\langle \mathbf{X}, \pounds_{[U,V]}\mathbf{Y}\big\rangle\,. \end{aligned}$$

In the last line, we referred to equation (3.a.3) and expanded it using the Leibniz rule. Almost a footnote-comment: the last two lines do not precisely prove our required theorem as they stand: to identify the two pieces of each sum we need one small trick—replace one of $\mathbf{X}$ or $\mathbf{Y}$ with the basis one-forms or vector fields, and the result for the other becomes manifest.

Therefore, because equation (3.a.1) holds for all one-forms and vector fields, our induction work proves that it must be true for all tensor fields of arbitrary rank. ὅπερ ἔδει δεῖξαι

[2]You should probably suspect this is overkill: the induction step seemed to make no obvious use of the fact that $r, s \geq 1$. And, as shown below, the identity is almost trivially true for the case of scalars. Nevertheless, it is better to be over-precise than incorrect. (In the famous words of Blaise Pascal to a mathematician friend: “I have made this letter longer because I have not had the time to make it shorter.”)

Problem 4

The torsion and curvature tensors are defined respectively,

$$T(X,Y) = \nabla_X Y - \nabla_Y X - \pounds_X Y \qquad \text{and} \qquad R(X,Y) = \nabla_X\nabla_Y - \nabla_Y\nabla_X - \nabla_{[X,Y]}. \tag{4.a.1}$$

We are to prove

a) $T(fX, gY) = fgT(X,Y)$,

b) $R(fX, gY)hZ = fghR(X,Y)Z$,

for arbitrary functions f, g and h, and vector fields X, Y and Z.

Theorem a: $T(fX, gY) = fgT(X,Y)$.

proof: In both of the required proofs, we will make repeated uses of the 'defining' properties of the covariant derivative ∇ and of the Lie derivative. In particular, we will need the following properties of the connection:

(1) $\nabla_X Y$ is a tensor in the argument X. This means that as an operator, $\nabla_{fX+gY} = f\nabla_X + g\nabla_Y$.

(2) $\nabla_X Y$ obeys the Leibniz rule in Y. Specifically, this means $\nabla_X(fY) = X(f)Y + f\nabla_X Y$. This implies that $\nabla_X Y$ is linear in Y—which follows when f is a constant.

We are almost ready to 'prove the identity by brute force in a couple of lines.' Let's just prepare one more trick up our sleeve: we will need

$$\begin{aligned}[fX, gY] = \pounds_{fX}(gY) &= g\pounds_{fX}Y + \left(\pounds_{fX}g\right)Y,\\ &= -g\pounds_Y(fX) + fX(g)Y,\\ &= -gfY(X) - gXY(f) + fX(g)Y.\end{aligned}$$

Let us begin:

$$\begin{aligned}T(fX, gY) &= \nabla_{fX}(gY) - \nabla_{gY}(fX) - \pounds_{fX}(gY)\,,\\ &= f\nabla_X(gY) - g\nabla_Y(fX) - \pounds_{fX}(gY)\,,\\ &= fX(g)Y + fg\nabla_X Y - g(Y(f))X - fg\nabla_Y X + gfY(X) + gXY(f) - fX(g)Y,\\ &= fg\nabla_X Y - fg\nabla_Y X - gf\pounds_X Y,\end{aligned}$$

$$\therefore T(fX, gY) = fgT(X,Y). \tag{4.a.2}$$

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Theorem b: $R(fX, gY)hZ = fghR(X,Y)Z$.

proof: We have already collected all of the properties and identities necessary to straightforwardly prove the theorem. Therefore, we may proceed directly.

$$\begin{aligned}R(fX, gY)hZ =& \Big\{\nabla_{fX}\nabla_{gY} - \nabla_{gY}\nabla_{fX} - \nabla_{[fX,gY]}\Big\}hZ,\\ =& \Big\{f\nabla_X\left(g\nabla_Y\right) - g\nabla_Y\left(f\nabla_X\right) - fg\nabla_{[X,Y]} - f\nabla_{YX(g)} + g\nabla_{XY(f)}\Big\}hZ,\\ =& \Big\{fX(g)\nabla_Y + fg\nabla_X\nabla_Y - gY(f)\nabla_X - fg\nabla_Y\nabla_X - fg\nabla_{[X,Y]} - fX(g)\nabla_Y + gY(f)\nabla_X\Big\}hZ,\\ =& \Big\{fg\nabla_X\nabla_Y - fg\nabla_Y\nabla_X - fg\nabla_{[X,Y]}\Big\}hZ,\\ =& fgR(X,Y)hZ,\\ =& fg\Big\{\nabla_X\Big(Y(h)Z + h\nabla_Y Z\Big) - \nabla_Y\Big(X(h)Z + h\nabla_X Z\Big) - [X,Y]\,(h)Z - h\nabla_{[X,Y]}Z\Big\},\\ =& fg\Big\{\nabla_X\Big(Y(h)Z\Big) + X(h)\nabla_Y Z + h\nabla_X\nabla_Y Z - \nabla_Y\Big(X(h)Z\Big) - Y(h)\nabla_X Z - h\nabla_Y\nabla_X Z\\ & \qquad - X\left(\nabla_Y h\right)Z + Y\left(\nabla_X h\right)Z - h\nabla_{[X,Y]}Z\Big\},\\ =& fg\Big\{hR(X,Y)Z + \left(\nabla_X\left(Y(h)\right)\right)Z - \nabla_Y\left(X(h)\right)Z - X\left(Y(h)\right)Z + Y\left(X(h)\right)Z\Big\},\end{aligned}$$

$$\therefore R(fX, gY)hZ = fghR(X,Y)Z. \tag{4.b.1}$$

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